

# 6. LOGS AND EXPONENTIALS

## §6.1. Powers

If  $x$  is any number, and  $n$  is a positive integer, we define the **power**  $x^n$  to mean  $x.x.x \dots x$  with  $n$  factors.

So  $x^1 = x$

$$x^2 = x.x$$

$$x^3 = x.x.x$$

If we list the powers of  $x$  we get a geometric sequence  $x, x^2, x^3, \dots$

Each time we add 1 to the **index** (that's the integer above the  $x$ ) we multiply the power by  $x$ . Working backwards, every time we reduce the index by 1 we divide by  $x$ .

So we should define

$$x^0 = 1$$

$$x^{-1} = \frac{1}{x}$$

$$x^{-2} = \frac{1}{x^2}$$

.....

**Theorem 1 (Index Laws):** For all integers  $m, n$  and all real numbers  $x, y$  we have:

$$(1) (x^m)^n = x^{mn};$$

$$(2) x^{-n} = \frac{1}{x^n};$$

$$(3) (xy)^n = x^n y^n;$$

$$(4) x^m x^n = x^{m+n}.$$

**Proof:** If  $m, n$  are positive it's just a matter of counting the numbers of factors. If one or both is zero these are easily checked. If one or both is negative it's not quite straightforward, though not really that difficult – just a bit tedious. So let's omit the proof shall we? 🙄

Fractional powers can be defined so as to fit in with the index laws. For example what is  $2^{1/2}$ ? Let's write  $2^{1/2} = y$ . Then  $y^2 = 2^{1/2} \cdot 2^{1/2} = 2^{1/2 + 1/2} = 2^1 = 2$ . So  $2^{1/2}$  is a square root of 2. Logically there is no reason at this stage why we couldn't define  $2^{1/2} = -\sqrt{2}$ , but if we did it would cause severe problems later on. So let's choose to define  $2^{1/2} = \sqrt{2}$ , the positive square root.

When it comes to defining  $2^{1/3}$  we'd have no choice. There's only one real cube root of 2 and we would have to define it to be  $2^{1/3}$  to fit in with the index laws.

If  $m, n$  are integers with  $n > 0$ , and  $x$  is a real number, we define  $x^{m/n}$  to be the positive  $n$ 'th root of  $x^m$ . In this way we'll have defined  $x^y$  for all rational numbers  $y$ . Of course we'd have to prove the index laws all over again, in this enlarged environment. It is not difficult to do this – just messy and tedious. Some proofs are really enlightening and should be gone through. Other proofs are best left unproved! Just so long as you realise that, if you were sceptical, that I could write out a proof for you.

## §6.2. What Do We Mean by $2^x$ ?

What does  $2^x$  mean for an irrational numbers? What, for example, does  $2^{\sqrt{2}}$  mean? The answer lies in the concept of limits. Although  $\sqrt{2}$  is irrational it can be approximated by rational numbers. If we write out the decimal expansion of  $\sqrt{2} = 1.41421356\dots$  we can take the sequence

1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, ...

Each of these is a rational number. We could write the sequence as:

$1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \frac{1414213}{1000000}, \dots$

Because all of these are rational numbers we can define:

$2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, 2^{1.4142}, 2^{1.41421}, 2^{1.414213}, \dots$

in terms of 10<sup>th</sup> roots of  $2^{14}$ , 100<sup>th</sup> roots of  $2^{141}$  and so on.

It can be shown that this sequence of real numbers will approach a limit. We define that limit to be the value of  $2^{\sqrt{2}}$ . There are many other sequences of rational numbers that approach  $\sqrt{2}$  and it can be shown that if we raise 2 to each of them we'll get a sequence that approaches the same limit as before.

So we define  $2^x$  as the limit of:

$2^{x_1}, 2^{x_2}, 2^{x_3}, \dots$

for any sequence  $x_1, x_2, x_3, \dots$  that approaches  $x$ .

This sounds all very technical. Anything to do with limits does get rather hard going. In practice if you had to draw the graph of  $y = 2^x$  you'd plot the points for certain rational values of  $x$  and join these points by a smooth curve.

In the same way we can define  $a^x$  for all real numbers  $a > 0$ .

Of course  $1^x$  equals 1 for all values of  $x$ .

### §6.3. Logs

Every positive number is a power of 2. For example  $2^{3.321928095}$  is very, very close to 10 and there's some number  $y$ , very, very close to 3.321928095 where  $2^y$  is exactly equal to 10. This number is called “the logarithm of 10 to the base 2”.



The number  $y$  for which  $2^y = x$  is called the **logarithm of  $x$  to the base 2**.

It's written  $y = \log_2 x$ .

**Example 1:** Find  $\log_2 128$ .

**Solution:** Since  $2^7 = 128$  it follows that  $\log_2 128 = 7$ .

**Example 2:** Find  $\log_2(1/8)$ .

**Solution:** Since  $\frac{1}{8} = 2^{-3}$  it follows that  $\log_2(1/8) = -3$ .

If  $b > 1$ , the **logarithm** of  $x$  to the base  $b$  is defined to be that power of  $b$  which exactly equals  $x$ .

It is written  $\log_b x$ .

A useful slogan to remember is:

**Logs are powers and powers  
are logs**

**Example 3:** Find  $\log_{10} 1,000,000$ .

**Solution:** One million is what power of 10? The answer is  $10^6$ . Logs are powers and powers are logs. So  $\log_{10} 1,000,000 = 6$ .

**Example 4:** Find  $\log_3 1$ .

**Solution:**  $1 = 3^0$  so  $\log_3 1 = 0$ .

In fact, since  $b^0 = 1$  for any base, the logarithm of 1 is zero, for any base.

**Example 5:** Find  $\log_5 0$ .

**Solution:** For what  $y$  is  $5^y = 0$ ? The answer is that there is no such  $y$  and so  $\log_5 0$  doesn't exist.

In fact, for any base  $b$ ,  $\log_b x$  is undefined if  $x \leq 0$ . That's because  $b^y$  is always positive. It can never equal zero or a negative number.

**Theorem 2:** For all  $b, c, x, y$  with  $b, c > 1$ :

$$(1) \log_b(xy) = \log_b x + \log_b y;$$

$$(2) \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y;$$

$$(3) \log_b(x^y) = y \cdot \log_b x;$$

$$(4) \log_b x = \frac{\log_c x}{\log_c b}.$$

**Proof:** These are mostly just the index laws turned around.

Let  $X = \log_b x$ ,  $Y = \log_b y$  and  $B = \log_c b$  Then  $b^X = x$ ,  $b^Y = y$  and  $c^B = b$ .

$$(1) xy = b^X \cdot b^Y = b^{X+Y}.$$

Hence  $\log_b(xy) = X + Y = \log_b x + \log_b y$ .

$$(2) \log_b x = \log_b\left[\left(\frac{x}{y}\right) y\right] = \log_b\left(\frac{x}{y}\right) + \log_b y.$$

$$(3) x^y = (b^X)^y = b^{Xy} \text{ so } \log_b(x^y) = Xy = y \log_b x.$$

$$(4) x = b^X = (c^B)^X = c^{BX}.$$

$$\therefore \log_c x = BX \text{ and so } \log_b x = X = \frac{\log_c x}{B} = \frac{\log_c x}{\log_c b} \cdot \text{👋😊}$$

Theorem 2 displays the reason why logarithms were first invented. Back in the 17<sup>th</sup> century if you wanted to multiply two large numbers you didn't have the convenience of calculators. To multiply 245961 by 28284 would require a substantial long multiplication. (At least

it was better than in previous centuries when it would all have to be done using Roman numerals!)

In 1514 Napier published the first table of logarithms. to base 10. To multiply two numbers you looked up their logs, added them, and then looked up a table of antilogarithms (this is simply a table of  $10^x$ . To divide, we subtract the logs and to find the  $n$ 'th root we simply divide the log by  $n$ .

The base that was chosen for these tables was  $b = 10$ . The advantage of this is that you could use them for decimal numbers without having to worry too much about the position of the decimal point.

**Example 6:**  $\log_{10} 276.3977 = \log_{10}(100 \cdot 2.763977) = 2 + \log_{10} 2.763977$ .

So tables were constructed giving just logarithms of numbers between 1 and 10. Logarithms of numbers smaller than 1 or larger than 10 can be easily expressed in terms of numbers in this range. Six figure logarithms would occupy a whole book. Four figure logarithms just needed two pages.

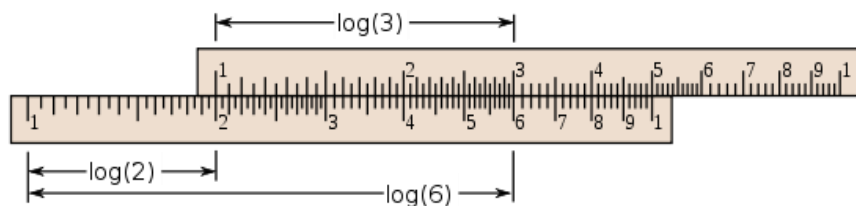
A mechanical version of log tables was invented a few years after Napier although the modern version was invented in the mid 19<sup>th</sup> century. This is called a 'slide rule'. In its most common form it consists of a strip of wood that slides in a slot in a wider strip of wood. There

are markings on both the base and the slider, representing numbers from 1 to 10. These are spaced according to their logarithms.

To multiply two numbers you place the 1 on the slider against the first number on the larger piece. Opposite the second number on the slider is the answer. You ignore



decimal points and work out where to put the decimal point in the answer. So multiplying 32.5 by 706 would use the same operation as multiplying 3250 by 0.00706. In both cases you'd get 229 and you would have to work out that this represented 22900 in the first case and 22.9 in the latter. The slide rule basically adds distances, and if the distances represent logarithms, they add logarithms.



With care slide rules could give about three significant figures in the answer. Not all 3 digit numbers are marked. One usually has to estimate the third figure by mentally subdividing the gap between numbers.



Slide rules have been made in many forms. There are circular slide rules and even cylindrical slide rules. A



cylindrical slide rule has the scale spiralling around and around giving a much greater effective length and therefore greater accuracy.

Up until the 1970's a slide rule was a symbol of the engineer. Just as a doctor is always seen with a stethoscope around her neck so an engineer in the 60's would always be seen with a slide rule sticking out of his pocket.

These days we don't need log tables or slide rules to multiply numbers. That makes them obsolete. But logarithms are useful for more than multiplying and dividing numbers. They are a vital tool in mathematics and there are many places where you need to know the value of a logarithm.

For example you've probably heard of the term 'decibel' as a measure of loudness. A 'bel', named after Alexander Graham Bell, represents a 10 fold increase in loudness, as measured by the power of the output. A decibel is one tenth of a bel. The scale is logarithmic in so far as a 100 fold increase in power is a 20 decibel increase in loudness. If  $R$  is the ratio of the power given out by two sounds then

the louder sound is  $\log_{10}R$  number of bels and so the louder sound is  $10\log_{10}R$  decibels louder than the other.

Another measurement that is based on logarithms to the base 10 is the Richter scale for earthquakes. If  $E$  is the amount of energy released by an earthquake then its magnitude on the Richter scale is  $\log_{10}E$ . So an earthquake that has magnitude 6.5 is 10 times more powerful than one that has magnitude 5.5 and is 100 times more powerful than one with magnitude 4.5.

## §6.4. Powers and Logs on Calculators

Most people know how to use calculators to multiply numbers on a calculator. Another thing we may need to do is to compute powers.

**$x^y$ :** For powers there's an  $x^y$  button. Use this, as follows.

(1) Input the base number  $x$ .

(2) Press the  $\boxed{x^y}$  button.

(3) Now input the power  $y$ .

(4) Press the  $\boxed{=}$  button.

The answer is now displayed.

**Example 7:** Find  $3.56^{7.29}$ .

**Solution:** 10472.98509.

There's a special number, called 'e' whose powers are quite important in mathematics – so much so that your scientific calculator will have a button for computing  $e^x$ .

This is called the **exponential function**. To understand the significance of  $e^x$  you'll need to know a little bit of calculus. Suffice to know that  $e$  is just a number, a bit like  $\pi$ , whose value can't be expressed as an exact decimal. Approximately  $e$  is 2.71828. In preparation for when you'll need to compute  $e^x$  here is how to use your calculator to compute it.

**$e^x$ :** You could use the  $x^y$  button but you'd have to input the base 2.71828. To save you the trouble most calculators have an  $e^x$  button.

(1) Input  $x$ .

(2) Press the  $e^x$  button.

On many calculators the  $e^x$  is not written on a button but is written *above* the "LN" button. To get  $e^x$  in such cases you have to press the  $\boxed{\text{INV}}$  key and follow it by the  $\boxed{\text{LN}}$  key.

If you can't find an the  $e^x$  key you should have a key marked  $\boxed{\text{exp}}$ . Use this instead.

The answer is now displayed.

**Example 8:** Find  $e^{2.9053}$ .

**Solution:**  $e^{2.9053} = 18.27072405$ .

Your calculator will have two log keys, one for base 10 and one for base  $e$ . If you want any other base you'll have to calculate it, using Theorem 2 (4).

There are three sets of notation that are used for these two bases. Look carefully at your calculator to see which one is yours.

	TYPE 1	TYPE 2	TYPE 3
<b>base 10</b>	$\log_{10}$	$\log_{10}$	LOG
<b>base e</b>	$\log_e$	log	LN

You can see the potential for confusion. On some calculators log means base 10 while on others it means base  $e$ . If you also have a key marked LN (standing for ‘natural logarithm’) then you know that LOG means base 10. Otherwise it will mean base  $e$ .

### **$\log_{10}x$ :**

(1) Input  $x$ .

(2) Press the  $\boxed{\log_{10}}$  key.

If you don’t have one of these try a key marked  $\boxed{\log}$  and use it instead.

(3) The answer is now displayed.

### **$\log x$ :** This is the natural logarithm, to the base ‘ $e$ ’.

(1) Input  $x$ .

(2) Press the  $\boxed{\text{LN}}$  key.

If you don’t have one of these look for a key marked  $\boxed{\log_e}$  and use this instead.

(3) The answer is now displayed.

If you ever want logs to other bases you can get them by using the formula:  $\log_b x = \frac{\log x}{\log b}$  or  $\frac{\text{LN } x}{\text{LN } b}$ .

**Example 9:** Find  $\log 16.8394$ .

**Solution:**  $\log 16.8394 = 2.82372139$ .

If you got 1.226326613 you must have pressed the ‘log’ key instead of the  $\log_e$  or LN key.

**Example 10:** Find  $\log_2 100$ .

**Solution:**  $\log_2 100 = \frac{\log_{10} 100}{\log_{10} 2} = \frac{2}{0.2010} = 6.438$ .

Of course we can use logs to the base  $e$  and get  $\log_2 100 = \frac{\text{LN } 100}{\text{LN } 2} = \frac{4.605}{0.6931} = 6.6438$ .

